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BALLISTIC RESEARCH LABORATORIES

REPORT No. 682

Steady State Solutions of the Equation of Burning

FRANZ L. ALT

ABERDEEN PROVING GROUND, MARYLAND

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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 682

Franz L. Alt
Aberdeen Proving Ground, Md.
1 September 1948

STEADY-STATE SOLUTIONS OF THE EQUATION OF BURNING

ABSTRACT

When a substance such as a low-order explosive or a propellant is ignited, the burning proceeds with a speed and temperature which are described by a certain partial differential equation. The general solution of this equation is not known. The present report deals with a special case of this equation, namely, the case of steady state, in which the burning progresses with uniform speed. We assume moreover that the burning substance has the shape of a thin rod of infinite length. The limitation to the steady-state case not only has the advantage of simplifying the problem mathematically, but it is interesting because in many practical cases the phenomenon of burning approaches the steady-state rapidly. Thus, the steady-state solutions presented in this report may also be thought of as limiting cases of more general solutions. Similarly the limitation to a thin infinite rod is made mostly because it simplifies the problem; but the solutions for many other shapes do not differ much from the ones obtained here.

The present report considers both the case of a perfectly heat-insulated rod and that of a rod which loses heat to its surroundings. For the first case, all possible solutions are obtained; for the second case, the report is limited to the commonly observed ranges of values of flame speed, heat loss and room temperature. Within these limits, the various possible types of solutions are discussed and a small number of solutions are presented numerically so that other solutions may be obtained by interpolation.

Sections 1 to 4 of this report contain the major results and an outline of the methods used in deriving them. Most of the detailed proofs as well as a few of the less interesting results have been relegated to the Appendix.

METHOD OF APPROACH

It is desired to find the changes of temperature in a material in which an exothermic chemical reaction, such as burning, is taking place. We consider the case in which the material has the shape of a thin rod of infinite length. Each point on the rod is characterized by its abscissa, X, measured in centimeters from some point as origin. The temperature T (measured in degrees absolute) in any point of the rod is a function of X and of the time t (measured in seconds from an arbitrary origin). Unless the temperature is the same for all X, heat will be conducted along the rod and will cause temperature changes which are subject to the well-known equation of heat flow,

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial X^2}$$

where k is a constant which depends on the thermic properties of the material. At the same time, heat will be generated in every point by the exothermic reaction. We assume[1] that the rate of increase in temperature brought about by this reaction is given by the Arrhenius expression $A\, e^{-Q/RT}$, where A and Q are constants determined by the nature of the reaction, and R is the gas constant. In addition to this rate of change in temperature and to the one due to heat conduction along the rod, we may have a certain loss of heat to the space surrounding the rod. We shall first consider the case in which the rod is perfectly heat-insulated; we then have

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial X^2} + A\, e^{-Q/RT} \tag{1}$$

This case is identical with that of a semi-infinite solid in which heat flows only inward from the surface and not parallel to the surface; and also with any other case of one-dimensional flow. Subsequently we shall deal with the case in which heat is lost to the surrounding space (or gained from it) by conduction. In this case

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial X^2} + A\, e^{-Q/RT} - C\,(T - T_s) \tag{2}$$

where we have assumed that the heat conduction between the rod and the surrounding medium is proportional to the difference in temperature (an assumption which is true as a first approximation). C is the factor of proportionality, and T_s is the temperature (assumed constant) of the space surrounding the burning rod.

[1] This assumption implies, somewhat unrealistically, that the burning substance remains chemically homogeneous throughout the process of burning.

We shall not attempt to find all solutions of the partial differential equations (1) and (2), but confine ourselves to steady-state solutions, that is those solutions for which T, as a function of X and t, remains constant along all lines

$$X - u\,t = \text{const.} \tag{3}$$

where u is a predetermined constant. In this case the problem of finding T for all X and t reduces itself to the simpler one of finding T for all X and one particular t, say t = 0; for

$$T(X, t) = T(X - ut, 0) \tag{4}$$

We shall write

$$X - ut = X^*, \; T(X - ut, 0) = T^*(X^*) \tag{4a}$$

The same facts may also be expressed by saying that "the points of constant temperature travel along the rod with uniform velocity u."

From (4) we have

$$\frac{\partial T(X, t)}{\partial X} = \frac{dT^*(X^*)}{dX^*}, \quad \frac{\partial T(X, t)}{\partial t} = -u\frac{dT^*(X^*)}{dX^*}$$

so that the partial differential equation (2) becomes an ordinary differential equation

$$k\frac{d^2T^*}{dX^{*2}} + u\frac{dT^*}{dX^*} + A\,e^{-Q/RT^*} - C\,(T^* - T_s) = 0\ . \tag{5}$$

Of the six parameters in this equation, three may be eliminated by a suitable choice of the units of measurement for X and T. If we set

$$\begin{aligned} &X^* = X - ut = x.\sqrt{kQ/RA}, \qquad T^*(X^*) = y(x)\cdot Q/R \\ &u = v\sqrt{kAR/Q}, \qquad C = c\cdot AR/Q, \qquad T_s = y_s\cdot Q/R \end{aligned} \tag{5a}$$

equation (5) becomes

$$\frac{d^2y}{dx^2} + v\,\frac{dy}{dx} + e^{-1/y} - c\,(y - y_s) = 0\ . \tag{6}$$

The symbols y and y_s represent absolute temperatures, expressed in units of Q/R. Similarly, v is the velocity of propagation of the burning process, and x may be considered as either time or distance, all measured in suitable units.

Equation (6) is the "equation of steady-state burning." In order to solve it we note that it does not contain x explicitly. Therefore, its order can be reduced by one. We consider y, rather than x, as the independent variable of the problem and set $dy/dx = -z(y)$. Then $dz/dy = (dz/dx) \cdot (dx/dy) = (d^2y/dx^2) \cdot (1/z)$, and (6) becomes

$$\frac{dz}{dy} = v - \frac{1}{z}\left(e^{-1/y} - c\,(y - y_s)\right) \qquad (7)$$

This is a non-linear ordinary differential equation of the first order. We shall refer to it as the "transformed equation." Our principal task will be to solve it; this is accomplished by numerical integration. Once this is done, the solution of (6) can be reduced to a simple quadrature, as follows: instead of finding y as a function of x, we find x as a function of y, from the equation

$$x = \int \frac{dx}{dy}\,dy = -\int \frac{dy}{z(y)} \qquad (8)$$

where $z(y)$ is the solution of (7). Having solved (8), we obtain the solution of (5) by changing the scale factors as in (5a), and the solution of (2) by means of (4).

Equations (5) and (6) are of the second order, and therefore two boundary conditions are required in order to determine a particular integral of either of these equations. One of these, which may be given in the form of prescribing the values of the temperature or its first derivative for a particular value of x, only serves to fix the location of the solution with respect to the x-axis. This is because, if one solution is given, infinitely many others may be obtained by shifting the former by an arbitrary amount parallel to the x-axis. The other boundary condition, however, has an essential influence upon the shape of the solution curve. It is customary, in dealing with the equation of burning, to prescribe that the process of burning should start from a non-burning equilibrium state: in the case without heat loss, the burning substance is supposed to have originally been at zero temperature (since at any other temperature it is not in equilibrium), and in the case with heat loss, it is supposed to have originally been at that temperature y_o which is just so much higher than the room temperature y_s that the heat generated within the substance, e^{-1/y_o}, is balanced by the heat lost to the surrounding space, $c(y_o - y_s)$. In other words, it is customary to prescribe, as the second boundary condition, that $y = 0$ (in the case $c = 0$) or $y = y_o$ (in the case $c \neq 0$) "a long time ago." It is not possible to postulate these conditions for any finite time (nor, by the same token, for any finite value of x), since this would contradict the assumption of a steady state. Therefore, the second boundary condition is formulated by stating that the temperature y should approach the limit stated (0 or y_o) as x tends to infinity, or, which is the same, as t tends to $-\infty$. Evidently, $z = -dy/dx$ must simultaneously tend to zero. Thus, in terms of the transformed equation, this boundary condition states that $z = 0$ for $y = 0$ (in the case $c = 0$) or for $y = y_o$ (in the case $c \neq 0$).

SOLUTIONS OF THE TRANSFORMED EQUATION WITHOUT HEAT LOSS

We consider first the case $c = 0$, i.e., the case of a perfectly insulated rod. We confine ourselves to non-negative values of y. This limitation suggests itself because of the interpretation of y as the absolute temperature of a substance (except for a scale factor). Figure 1 shows the shape of several solution curves of (7). Figures 1 - 4 are not drawn to scale. They merely show the general appearance of the solutions. In Figures 1, 2 and 4 the temperature (y) is plotted horizontally, and the rate of change of temperature (z) is plotted vertically.

There are infinitely many solution curves similar to curve I-A of the figure. They start at the origin with horizontal tangent, increase to a maximum, then decrease, intersect the y axis vertically, turn back in the direction of decreasing y and reach the z axis for a certain negative value of z; at the intersection with the z axis the slope is v. There is one and only one curve of this type through every point in the area bounded by the z axis and curve I-B.

Curve I-B is similar in shape to the curves of Type I-A except that its slope at the origin is v rather than 0. There is only one curve of this type. In the application of this differential equation to the problem of burning this is the most interesting solution.

The solutions of Type I-C are similar in shape to curve I-B but they start at a point on the z axis with some positive value of z, rather than at the origin. The slope at this point is again v. There are infinitely many solutions of this type. One and only one passes through each point in the area between curves I-B and II-C.

Solution II-C starts at the z axis with positive z and slope v, increases monotonically and approaches the value $z = 1/v$ asymptotically. There is only one solution of this type.

The solutions of Type III-C start at the z axis with a value of z larger than that of solution II-C. Their slope at the starting point is v. With increasing y they increase to infinity. Their slope decreases from y to a positive minimum value, then increases again and approaches v as y goes to infinity. One and only one solution of this type passes through each point in the area above curve II-B. The curves of Type I-C, II-C and III-C satisfy the differential equation but not the boundary condition $z(0) = 0$. On the other hand, all curves of Type I-A as well as the single curve of Type I-B are admissible solutions of our problem.

The five types of solution curves which we have described and which are sketched in Figure 1 exist if v is considerably smaller than one. If the value of v is increased, the starting point of solution II-C moves closer to the origin. For a certain value of v it coincides with the origin. For this v there are no solutions of Types I-B and I-C. We shall denote this limiting velocity by V. Its value has been determined by repeated numerical integration:

$$V = .90280.$$

If v increases beyond this value, then the solution curves have the shapes shown in Figure 2. The solutions of Type I-A are similar to those for small v. They fill the area below curve II-A. The latter is a solution curve which starts at the origin with slope 0 and increases asymptotically toward $z = 1/v$. Curves III-A, III-B and III-C of Figure 2 are similar to the curves III-C of Figure 1, except that III-A and III-B start at the origin and III-A has a horizontal slope at the origin. There is only one solution of Type III-B but infinitely many of Types III-A and III-C.

SOLUTIONS OF THE EQUATION OF STEADY-STATE BURNING WITHOUT HEAT LOSS

In the preceding paragraph we have discussed solutions of the transformed Equation (7). We now turn to the corresponding solutions of Equation (6) still limiting ourselves to the case of perfect insulation ($c = 0$). These solutions can be found by the simple integration indicated in Equation (8).

Solutions of Equation (6) corresponding to the five types of Figure 1 are shown in Figure 3. Note that on this diagram the temperature y is plotted vertically whereas in Figure 1 it was plotted horizontally.

The horizontal axis in Figure 3 may be interpreted in two ways: as the X-axis along the burning rod, where burning is assumed to progress from left to right; or as the time axis, where time is read from right to left. In the former interpretation, each of the curves shown in Figure 3 is a possible temperature distribution along the rod, "possible" meaning a distribution which will result in a steady flow of heat; that is to say, if one of these distributions exists initially, then the generation of heat within the material of the rod and the flow of heat along the rod will change the temperature in each point in such a way that after any interval of time the temperature distribution will appear to have been shifted to the right without having changed its shape. In the second interpretation, we may find out how the temperature in any desired point of the rod changes with time, by traveling along one of the curves from right to left. We see that "a long time ago" the temperature was zero (absolute), that it rises gradually, and that on some curves it reaches a maximum and declines again.

The solutions of Types I-C, II-C and III-C reach an absolute temperature of 0 at some finite point X (or, in the other interpretation, the temperature starts at 0 at some finite time). These solutions are, therefore, not representations of any steady-state phenomenon. Solutions of Type I-A or I-B, on the other hand, can be interpreted as temperature curves along a burning rod in the steady-state. On each of these curves y has a maximum which might be thought of as the temperature of the burning front.[1] This front should be visualized as proceeding along the burning rod from left to right (the direction of burning is determined by the sign of u in Equation (3)). The portion of the temperature curve to the left of the burning front (and correspondingly that portion of the solutions of (7) for which z is negative) covers that part of the rod which is already burned. Because of the chemical changes caused by burning, Equation (1) no longer represents the temperature changes in this region.

In Figure 3 all solution curves have been drawn through one point. That is to say, we assume that the temperature in that particular point of the rod is given. Other solutions are obtained by shifting those shown in the diagram to the left or right by any desired distance.

Certain points in Figure 3 are lettered the same as the corresponding points in Figure 1, to facilitate comparison.

The maximum ordinate of curve I-B is of particular interest. It represents the highest burning

[1] Actually this temperature will usually differ from the observed burning temperature because the conditions prevailing in a burning substance are not accurately described by Equation (1), especially not at high temperatures at which most burning substances undergo chemical changes. It is probable that a point somewhat to the right of the maximum in Figure 3 corresponds more closely to the actual burning front.

temperature which could possibly be reached in steady-state burning. Its value depends on v. The following values were determined by numerical integration:

v	Maximum y
10^{-1}	0.2
10^{-3}	0.0692
2×10^{-4}	0.0565
2×10^{-5}	0.0448
2×10^{-8}	0.0277

These values are plotted in Figure 5. The accuracy of the last decimal place is doubtful. The figures can be converted to customary units, (degrees, centimeters, etc.) by means of Equations (5a). As v grows beyond 0.1 toward the value V = .90280, the maximum of y grows rapidly to infinity.

The only difficulty in the integration process (8) arises when z becomes 0. This occurs at most twice on each solution curve. It can be shown that at the points where the solution of (7) intersects the y axis with vertical slope, the integral of 1/z has a finite value. On the other hand, for those solutions of (7) which pass through the origin, the integral of 1/z goes to infinity as y tends to 0.

SOLUTIONS ALLOWING FOR HEAT LOSS

We now turn to the case in which c has some positive value, that is the case in which heat is lost to the medium surrounding the burning substance. The shapes of the solution curves in this case vary greatly with the varying values of v, c and y_s. In this discussion we shall confine ourselves to those values which are found in actual experiments with burning substances. The range for v is that shown in the table in the preceding section. Realistic values of c are of a magnitude between 10^{-15} and 10^{-22}. For y_s, which represents room temperature, the values 0.012 and 0.017 are used. The difference between these values is caused not so much by variations in real room temperature, as by the differences in the constant Q, which determines the scale in which y is measured.

Furthermore, this discussion will be confined to that range of y which corresponds to temperatures commonly measured. An added reason for this limitation is that for extreme values of y the heat loss is probably not adequately represented by the term $c(y - y_s)$. Some indication of the behavior of solutions in this extreme range is given in the Appendix.

The shape of the solution curves of (7) is shown in Figure 4. In this figure y_0 represents the temperature of the rod before burning, y_1 the ignition temperature. The value y_0 is slightly larger than room temperature. This is because even at room temperature some heat is generated in the combustible substance by the chemical reactions taking place. The difference in temperature between the rod and the surrounding space must be just large enough so that the small amount of heat generated equals the heat lost to the surrounding space. As the temperature rises above y_0, the heat loss increases at first more rapidly than the amount of heat generated. For still higher temperatures, however, the latter increases more rapidly; when the temperature reaches the value y_1, heat loss and heat generation are again in balance.

As before, we shall classify the solutions according to their behavior with increasing y. Some reach a maximum and then decrease to $z = 0$, (Type I), some grow to infinity (Type III), and one approaches asymptotically the value $z = 1/v$ (Type II). The further subdivision according to shape near the singular points differs somewhat from the case without heat loss.

In Figure 4, to the right of y_1 the shape of the curves is similar to those in Figure 1. However, the slope of Type I-A curves at the point y_1 is not 0 but a small positive constant. The slope of curve I-B at the point y_1 is slightly smaller than v. All other curves intersect the line $y = y_1$ with slope v.

To the left of y_1 there are infinitely many solution curves through the point $y = y_1$, $z = 0$. All of them have the same small positive slope, except for one which has a slope slightly smaller than v. The latter is the continuation of curve I-B, the former are the continuations of the curves of Type I-A. Of these, one passes through the point $y = y_0$, $z = 0$, while the others are above or below it.

In order to have our boundary conditions satisfied, we have to consider solutions in the (y, z) plane through the point $y = y_0$, $z = 0$. There are only two solution curves through this point, one increasing and one decreasing. This fact is of considerable importance in the physical interpretation of the solutions. The decreasing curve represents not a burning substance but rather a substance in which the temperature is initially at the ignition point and drops slowly toward room temperature. (In Figure 4 this is the curve of Type I-A mentioned above; with a different selection of parameters it might be a curve of different type.) The other of the two curves passing through the point y_0 is marked "I-D" in Figure 4. It represents steady-state burning. It is interesting to note that the latter solution is unique. On the other hand, in the case without heat loss ($c = 0$) we saw that there are infinitely many solutions which start at 0 temperature and represent steady-state burning.

The infinitely many solution curves passing through the point y_1 represent substances in which the temperature was initially at the ignition point. Since this is a point of unstable equilibrium, it is not surprising to find that there are infinitely many ways in which the temperature, starting from this value, can either increase or decrease.

Figure 4 shows the field of the transformed equation (7), similar to Figures 1 and 2. It is easy to visualize the shape of the corresponding solutions of the equation of burning, in analogy to Figure 3. In particular, the solution corresponding to I-D in Figure 4—the only one which satisfies our boundary conditions—looks like I-B in Figure 3, except that its asymptote on the right is not zero but y_0.

It is of primary interest to determine numerically the maximum temperatures for the unique solution of steady-state burning. The following values were obtained by numerical integration of Equation (7):

v	y_s	Maximum y
10^{-3}	.012	.0672
10^{-3}	.017	.0663
2×10^{-5}	.017	.0428
2×10^{-8}	.012	.0268

These figures, as well as those given in "Solutions of the Equation of Steady-State Burning without Heat Loss," previously, are plotted in Figure 5. The accuracy of the last decimal place is in doubt. The difference between the values of y given in the two tables is approximately 3% in the case where y_s is .012, and 4% where y_s is .017. These percentages can certainly not remain constant for all values of v, but in the range covered by this report they represent a sufficiently good approximation.

Note that the value of c is not listed in the above table. This is because the results are practically independent of c as long as c is small but positive. The presence of c does, however, have the measurable effect of "stopping" the solution curves at $y = y_0$, rather than at $y = 0$.

The temperature curves which have been calculated numerically are shown in the attached graphs 9-13. Those for $c = 0$ are solutions of Type I-B, those for positive c are solutions of Type I-D. Figures 9-12 are plots of the temperature gradient against temperature, similar to Figures 1 and 4. Figure 13 shows the temperature distribution along the burning substance, similar to Figure 3. One difference between Figures 3 and 13 is that the former shows all integral curves going through one point, whereas in the latter each curve is assumed to reach its maximum for $x = 0$. This is, of course, a matter of choice of initial conditions. Moreover, Figure 13 has a logarithmic scale for x, so that $x = 0$ is not shown.

APPENDIX

Computational Procedure. In the application of Equations (1) and (2) to steady-state burning, the most interesting solutions are (a) in the case $c = 0$, the solution of Type I-B, and (b) in the case of positive c, the solution of Type I-D. In the form (7) the equation can be integrated numerically by one of the usual methods. Only the points for which $z = 0$ need special attention, since in these points the right-hand side of (7) is not determined. These points shall now be considered.

We shall use the abbreviations

$$p(y) = e^{-1/y} - c\,(y - y_s) \tag{9}$$

$$q\,(y, z) = v - \frac{p(y)}{z} \tag{10}$$

so that Equation (7) can be written

$$\frac{dz}{dy} = q\,(y, z) \tag{11}$$

The points in which the integrals of this equation intersect the y-axis are of two types: those where $p(y) \neq 0$, and those where $p(y) = 0$.

In the former case, if we approach a point $z = 0$ along an integral curve of (11), dz/dy increases beyond all bounds. The process of numerical integration can nevertheless be continued in the neighborhood of these points by changing variables, considering z as the independent and y as the dependent variable, and integrating the equation in the form

$$\frac{dy}{dz} = \frac{1}{v - \frac{1}{z}\left(e^{-1/y} - c\,(y - y_s)\right)}$$

It remains to investigate the points where $z = 0$ and $p(y) = 0$. It can be shown by elementary considerations that the function $p(y)$ vanishes in one, two or three points (for real positive y) depending on the values of the constants c and y_s. If c is larger than or equal to $4\,e^{-2}$, then p vanishes in one place only, regardless of the value of y_s. If c is less than $4\,e^{-2}$, then p vanishes in three places provided y_s lies between certain boundaries. For c less than e^{-1}, the lower of these boundaries is zero, i.e., p vanishes in three places provided only that y_s is small enough. In practice, c is always very much smaller than either of the two limits mentioned, being of a magnitude between 10^{-15} and 10^{-22}. Moreover, in all practical cases y_s is so small that $p(y)$ vanishes for three different values of y. We shall denote these three places in order by y_0, y_1, y_2. The smallest, y_0, is always slightly larger than y_s; it is the temperature of the combustible material before burning has begun. y_1 is the ignition temperature. y_2 is, in the usual applications, so large that it is never reached.

If $c = 0$, then $p(y)$ vanishes only for $y = 0$.

We shall see below that if an integral curve of (11) passes through the point $(y_1, 0)$, then in that point its slope has one of the two values

$$\frac{v}{2} \pm \sqrt{\frac{v^2}{4} - (\frac{1}{y_1^2} e^{-1/y_1} - c)} .$$

If an integral curve of (11) passes through the point (y_0, 0), then its slope in that point has one of the two values

$$\frac{v}{2} \pm \sqrt{\frac{v^2}{4} - (\frac{1}{y_0^2} e^{-1/y_0} - c)} ;$$

in practical cases, these two values are almost equal to $v + \frac{c}{v}$ and to $-\frac{c}{v}$, respectively. If, in the case $c = 0$, an integral curve passes through the origin, then its slope at the origin has one of the two values v or 0.

For the numerical computation it is important to remember that the integral curve going through the origin with slope v, or through (y_0, 0) with slope $v + c/v$, remains almost straight for a considerable distance. Thus, the numerical integration along these curves can be begun with fairly large steps. At some distance from the origin or the point (y_0, 0), the slope begins to differ noticeably from its initial value, and then the length of the integration step must be diminished.

The solution of Equation (6) is obtained, as explained in "Method of Approach" (page 6), by integrating the differential $-\frac{dy}{z}$. However, the integrand becomes infinite when z is zero. If simultaneously p(y) is different from zero, then the integration can nevertheless be carried out; for in the neighborhood of such a singularity

$$\frac{dy}{z} = \frac{1}{z} \cdot \frac{1}{\frac{dz}{dy}} \cdot dz = \frac{dz}{z\,(v - \frac{p(y)}{z})} = \frac{dz}{z\,v - p(y)}$$

and in the last form the denominator differs from zero even at the singular point. On the other hand, if p(y) also vanishes, then it can easily be seen that the integral becomes infinite. In other words, corresponding to those solutions of (7) which go through the origin or through the point (y_0, 0) or (y_1, 0), we have solutions of (6) which approach asymptotically, as x goes to infinity, the value $y = 0$ or $y = y_0$ or $y = y_1$.

Integral curves in the singular points ($z = p(y) = 0$). (a) In the case $c \neq 0$, the behavior of the integral curves of (7) or (11) can be determined by known methods (see, for example, E. Goursat, Course in Mathematical Analysis, Vol. II, Part II, pp. 179-180). If (y^*, 0) represents one of the three singular points (y_0, 0), (y_1, 0), (y_2, 0) of Equation (11), the integral curves of (11) in the neighborhood of this point are known to be similar to those of the equation

$$\frac{dz}{dy} = \frac{v\,z - (y - y^*) \cdot p'(y^*)}{z} .$$

Integral curves of the latter equation have at (y^*, 0) one of the two slopes

$$s' = \frac{1}{2}v + \sqrt{\frac{1}{4}v^2 - p'(y^*)}$$

and

$$s'' = \frac{1}{2}v - \sqrt{\frac{1}{4}v^2 - p'(y^*)} \quad .$$

If s', s'' are real and have opposite sign (i.e., if p' is negative, which is the case for y_o and y_2 but not for y_1), then there are only two integral curves passing through the singular point; the latter is called a "Saddleback". (See the point y_o in Figure 4). At $(y_1, 0)$, p' is positive, so that s', s'' are either real and positive or imaginary. If they are real, the point is a node; through it pass one integral curve with slope s' and infinitely many with slope s''. If s', s'' are complex (or real but equal), then the point $(y_1, 0)$ is a focus, the integral curves have the form of spirals which approach the focus. The corresponding solutions of equation (6) give a temperature which oscillates above and below the ignition point y_1. The condition for the existence of such oscillating solutions is

$$p'(y_1) > \frac{1}{4}v^2 .$$

(b) In the case $c = 0$ the above method is not applicable because p' vanishes for $y = 0$ (which is the only singular point in this case). We can, however, proceed as follows:

Suppose that a solution of (7) or (11) goes through the origin, and has in that point a derivative $z'(0)$. Then

$$z'(0) = \lim_{y=0} \frac{z(y)}{y} \quad .$$

According to a theorem about real functions, there is a sequence of points y, converging to 0, for which

$$z'(0) = \lim_{y=0} z'(y) \; .$$

Thus

$$z'(0) = \lim_{y=0} \left(v - \frac{p(y)}{z}\right) = v - \lim_{y=0} \left(\frac{p(y)}{y} \cdot \frac{y}{z}\right) = v - \lim_{y=0} \frac{p(y)}{y} \cdot \lim_{y=0} \frac{y}{z} \quad . \tag{12}$$

The last step, however, is permissible only if we know that at least one of the two limits on the right converges to a value different from zero. Now if $z'(0) \neq 0$, then

$$\lim_{y=0} \frac{y}{z}$$

exists and is equal to $1/z'(0)$, and therefore different from 0, and it follows that the last equation of (12) holds. Its only solution is $z'(0) = v$. Thus, $z'(0) = 0$ and $z'(0) = v$ are the only possible slopes of integral curves of (7) at the origin in the case $c = 0$.

We shall prove in the next section that the origin is a node; there is only one integral curve of (7) going through the origin with slope v, while there are infinitely many with slope 0. This proof is based on a study of the curvature of the integral curves.

The Curvature of Integral Curves. In the further discussion we shall denote, as before,

$$p'(y) = \frac{dp(y)}{dy} \tag{13}$$

and furthermore by q' the derivative of q with respect to y in the direction of the integral curve of (7):

$$\begin{aligned} q'(y, z) &= \frac{\partial q(y, z)}{\partial y} + \frac{\partial q(y, z)}{\partial z} \cdot \frac{dz}{dy} \\ &= -\frac{p'(y)}{z} + \frac{p(y)}{z^2} \cdot q(y, z) \quad . \end{aligned} \tag{14}$$

An integral curve going through the point (y, z) is convex or concave in that point depending on the sign of $q'(y, z)$. In the (y, z) plane the region where q' is positive is separated from the region where it is negative by a curve along which $q' = 0$ (and possibly by portions of the y-axis, on which q' is not defined).
For this curve we have

$$0 = q' = -\frac{p'}{z} + \frac{p}{z^2} \cdot q = -\frac{p'}{z} + \frac{vp}{z^2} - \frac{p^2}{z^3}$$

or

$$p' \cdot z^2 - vp \cdot z + p^2 = 0 \quad . \tag{15}$$

This equation, which determines z as a function of y, is the equation of the curve on which $q' = 0$. The discriminant of this quadratic equation,

$$D = p^2 (v^2 - 4p')$$

vanishes if either $p = 0$ or $v^2 = 4p'$. It is important to find the places where $D = 0$ because Equation (15) gives two, one or no value of z for a given y depending on the sign of D for that y. Now p^2 vanishes in the places y_0, y_1 and y_2 in the case $c \neq 0$, and for $y = 0$ only in the case $c = 0$; everywhere else p^2 is positive. On the other hand, the sign of $v^2 - 4p'$ depends on the value of v and c. The function $p'(y)$ has a maximum at $y = 1/2$; its value is $p'(\frac{1}{2}) = 4e^{-2} - c$. From this value, p' decreases to both sides, vanishes in two places which we shall call y_1^* and y_2^*, and approaches the value $-c$ as y approaches 0 or infinity. Thus, if v is larger than

$$\sqrt{16 e^{-2} - 4c},$$

then $v^2 - 4p'$ is always positive, and D is positive everywhere except where $p = 0$. In this case the curve $q' = 0$ consists of two branches which are separate for all positive values of y, except y_0, y_1, y_2. In the two places where $p' = 0$, Equation (15) becomes linear and has only one solution, namely, $z = p/v$; one of the two branches of the curve $q' = 0$ goes to $z = \infty$. The shape of the curve is as shown in Figure 6. (This figure as well as Figures 7 and 8 are not drawn to scale; they merely show the general appearance of the curves.)

If, however, v is smaller than $\sqrt{16\,e^{-2} - 4c}$, then $v^2 - 4p'$ is zero in two points which lie between y_1^* and y_2^* and to different sides of $y = 1/2$. Between these two points D is negative. If this interval excludes the point y_1, i.e., if $D(y_1)$ is positive, then the shape of the curve $q' = 0$ is as shown in Figure 7. The shape of this curve is further modified in an obvious way if $D(y_1)$ is negative, or if it is positive but y_1 is larger than 1/2. In Figure 7, the sign of $q'(y, z)$ is indicated by + and - signs. In the + regions the integral curves of Equation (7) are curved upwards, in the - regions they are curved downwards. Compare this diagram with Figure 4 where some of the integral curves are shown.

Figure 8 shows the types of curves $q' = 0$ which can occur in the case $c = 0$ for various values of v. The dotted line corresponds to $v = 4/e$, for which D vanishes only at $y = 1/2$ and $y = 0$.

Let us now calculate the derivative of $q'(y, z)$ with respect to y in the direction of the integral curves of (7). This is

$$q''(y, z) = \frac{\partial}{\partial y}\, q'(y, z) + \frac{\partial}{\partial z}\, q'(y, z) \cdot q(y, z) \quad .$$

An easy transformation shows that

$$q''(y, z) = -\frac{p''(y)}{z} - \left(\frac{2v}{z} - \frac{3p}{z^2}\right) q'(y, z) \quad .$$

Now $p''(y)$ is positive between $y = 0$ and $y = 1/2$, and negative for all larger values of y. If we consider a point on the curve $q' = 0$, then $q'' = -p''/z$ is positive for $y < 1/2$, $z < 0$, and for $y > 1/2$, $z > 0$; q'' is negative for $y < 1/2$, $z > 0$ and for $y > 1/2$, $z < 0$. From this fact, a number of conclusions can be drawn concerning the shape of the integral curves of (7). For instance, consider the case of Figure 8 for small v. The integral curves of Equation (7) have positive curvature ($q' > 0$) inside the loop which is formed by the left-hand part of the curve $q' = 0$, as well as to the right of the right-hand part of that curve. If an integral curve passes through a point inside the loop, then the entire portion of the integral curve to the left of that point is also within the loop. In other words, an integral curve can leave the loop only in the direction of increasing y, but not in the direction of decreasing y. For in a point where the integral curve crosses the boundary of the loop, q' is zero and q'' is negative, therefore to the right of this point on the integral curve, q' is negative, and thus the points on the right are outside the loop; that is to say, in crossing the boundary of the loop in the direction of increasing y along an integral curve of Equation (7), we are leaving the loop, not entering it. Similarly, an integral curve of (7) can only enter but not leave, in the direction of increasing y, the area to the right of the right-hand part of the curve $q' = 0$. Analogous conclusions can be drawn about the various portions of the area of positive q' in the cases of Figure 6 and Figure 7.

A further conclusion, in the case of integral curves going through points inside the loop in Figure 8, is that all such curves will go through the origin, and that in the neighborhood of the origin their curvature remains positive. Their slope at the origin is 0. There are infinitely many curves of this kind, since they fill the entire area of the loop. They are the curves of "Type I-A" mentioned in "Solution of the Transformed Equation without Heat Loss," of this report. A similar reasoning applied to the right-hand portion of the area where q' is positive, leads to the curves of "Type III-C."

SUMMARY

Steady-state burning, as defined by Equation (5) or (6) can take on one of the following forms:

a. In the case without heat loss to the surrounding space, there are infinitely many solutions in which the temperature approaches absolute zero asymptotically as x tends to infinity, has a maximum at some finite x and decreases toward zero to the left; one of these is distinguished by the fact that its maximum temperature is higher than that of any other solution in this group. There are also infinitely many solutions (these, however, do not fulfill the boundary conditions of steady-state burning) in which the temperature reaches zero for a finite x and increases beyond all bounds as x decreases. Between these two groups of solutions there is a third whose character depends on the value of the burning velocity, v. If this is smaller than .90280, the solutions in this group have a maximum temperature for some value of x, and the temperature declines toward absolute zero to both sides of the maximum. If v is greater than, or equal to, .90280 then the temperature in this intermediate group of solutions approaches infinity as x decreases, and approaches zero as x grows to infinity. The limiting case of the first group appears to be the most interesting one, and several examples (for various values of v) are plotted on Chart 13. However, all the solutions of the first group are consistent with the conditions of steady-state burning.

b. In the case with heat loss, there is a unique solution which is consistent with the conditions of steady-state burning. On it, the temperature reaches a maximum for some finite x, declines to both sides and approaches the "initial temperature" y_0 (close to room temperature) asymptotically as x grows to infinity. (For declining x, the temperature declines to absolute zero; but this part of the curve no longer represents burning in the same substance.) Examples of such solutions, for varying values of flame speed (v) and room temperature (y_s) are shown in Chart 13. The solutions are practically independent of the value of c, the constant which determines the amount of heat loss, as long as c is reasonably small but different from zero.

There are numerous other solutions of the differential equation of steady-state burning with heat loss, but none of them satisfies the initial conditions usually associated with burning; in particular, there is no other solution which starts at room temperature. There are infinitely many solutions which approach the ignition temperature y_1 from above or below as x increases to infinity (curves of Type I-A and I-C, respectively); others are similar to the solutions of Type II-C and III-C of the case without heat loss in that the temperature increases to infinity with decreasing x; a few solutions (shown in Diagram 4 near the origin) remain at low temperatures throughout.

Chart 4 shows all types of solutions (of the "transformed equation") for the case in which the parameters v, c, y_s have realistic values; in particular, that v is smaller than V, that y_s is sufficiently small so that p(y) has three real roots, and that c is small. Other types of solutions occur for values of the parameters outside these limits. For instance, if c is large enough relative to v, so that $p'(y_1) > \frac{1}{4}v^2$, then the transformed equation has solutions which spiral around the point y_1, and the corresponding solutions of the equation of burning oscillate about the ignition temperature.

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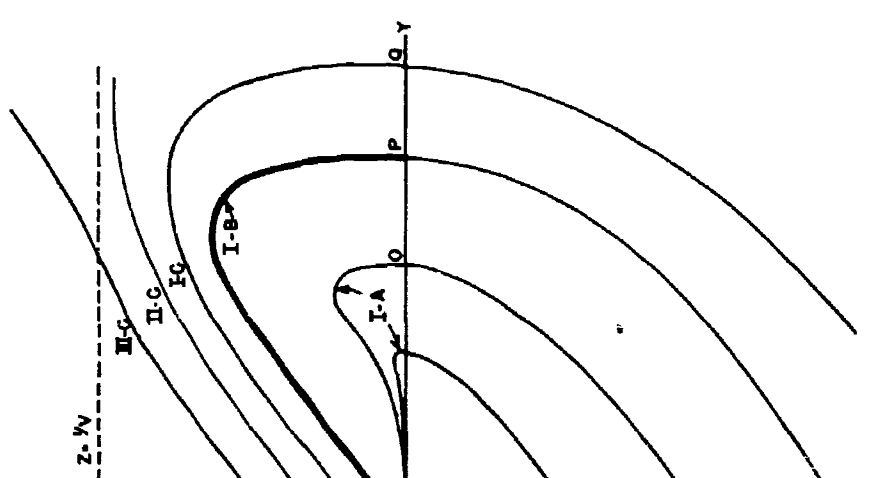


FIG. 1 - SHAPE OF SOLUTIONS OF THE TRANSFORMED EQUATION. NO HEAT LOSS - V<.90280 TEMPERATURE GRADIENT (Z) VS. TEMPERATURE (Y)

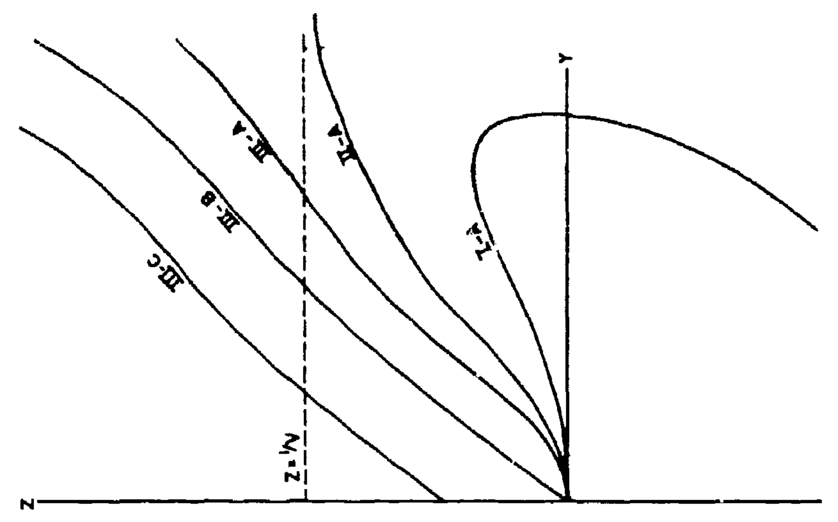
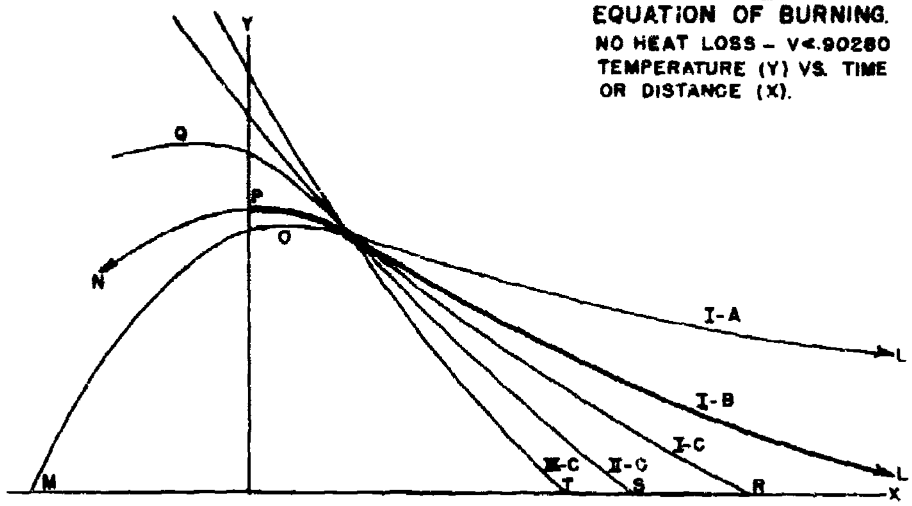


FIG. 2 - SHAPE OF SOLUTIONS OF THE TRANSFORMED EQUATION. NO HEAT LOSS - V>.90280 TEMPERATURE GRADIENT (Z) VS. TEMPERATURE (Y)

FIG. 3 - SHAPE OF SOLUTIONS OF THE EQUATION OF BURNING.
NO HEAT LOSS - V<.90280
TEMPERATURE (Y) VS. TIME OR DISTANCE (X).



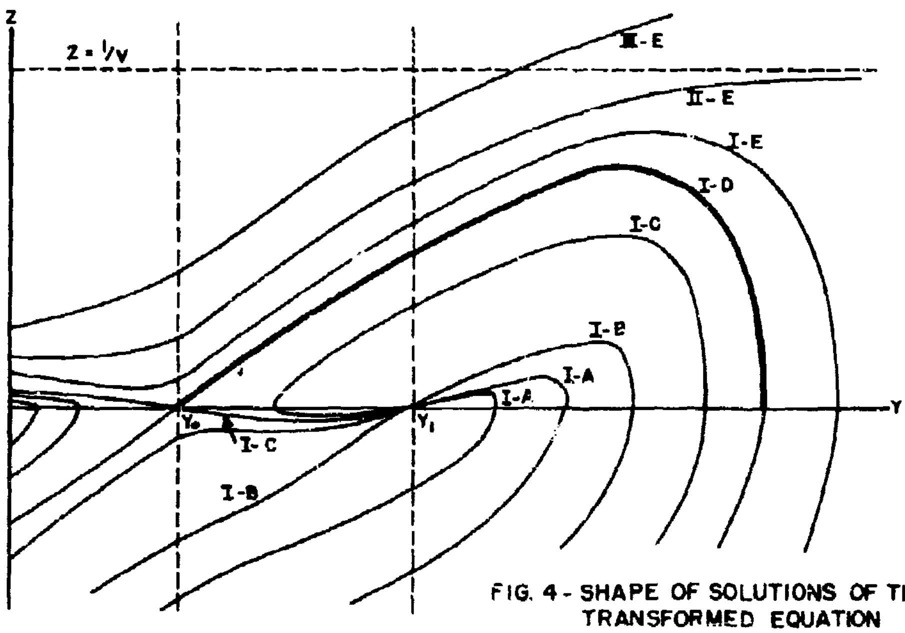


FIG. 4 - SHAPE OF SOLUTIONS OF THE TRANSFORMED EQUATION
WITH HEAT LOSS - TEMPERATURE GRADIENT (Z) VS TEMPERATURE (Y)

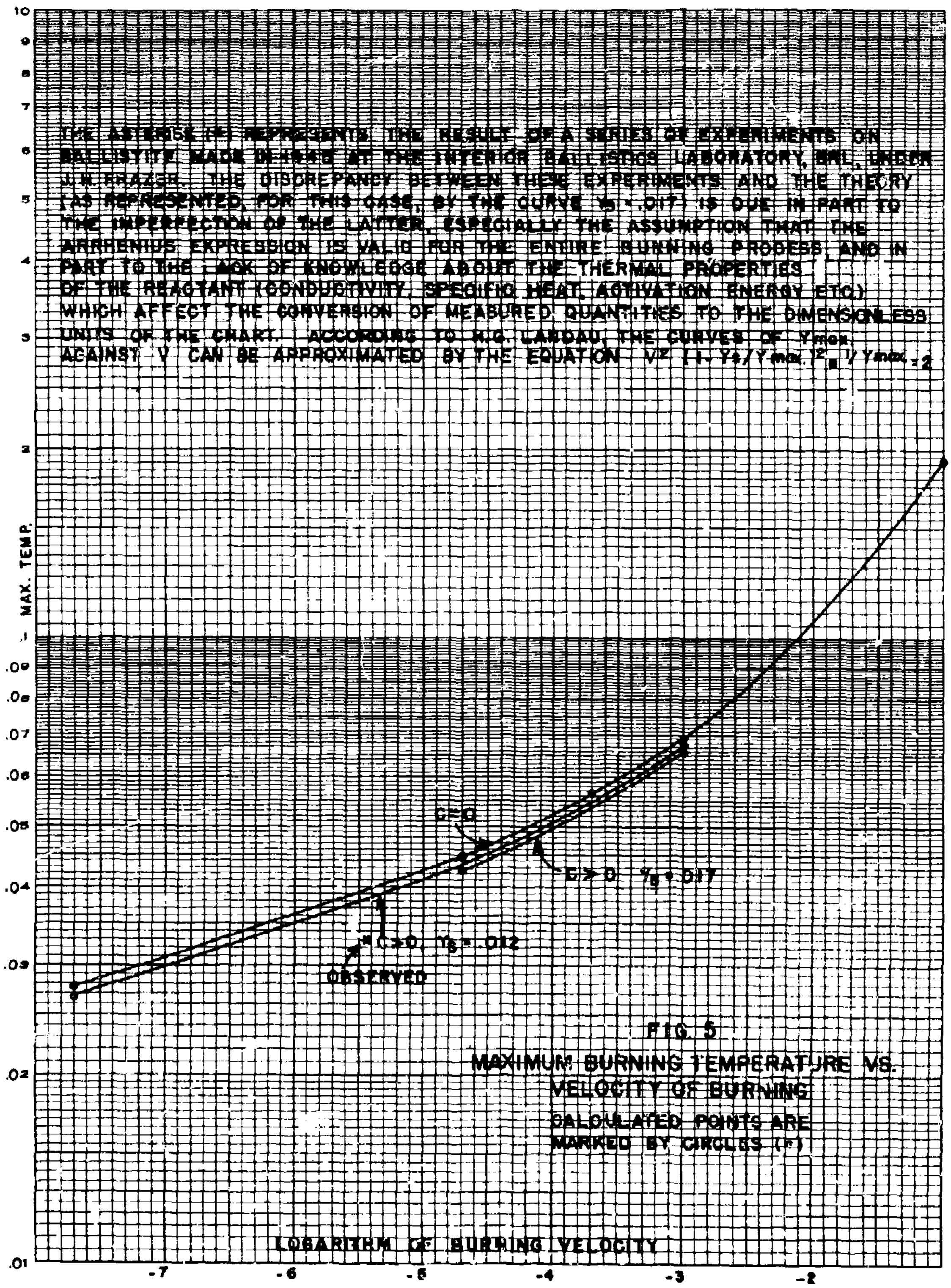


FIG. 5

MAXIMUM BURNING TEMPERATURE VS. VELOCITY OF BURNING

CALCULATED POINTS ARE MARKED BY CIRCLES (°)

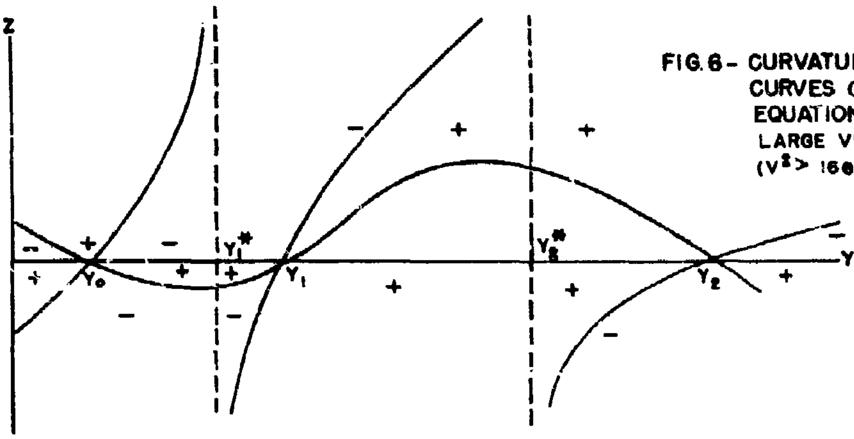


FIG. 6 - CURVATURE OF INTEGRAL CURVES OF THE TRANSFORMED EQUATION. WITH HEAT LOSS— LARGE VELOCITY OF BURNING ($V^2 > 16e^{-2} - 4c$)

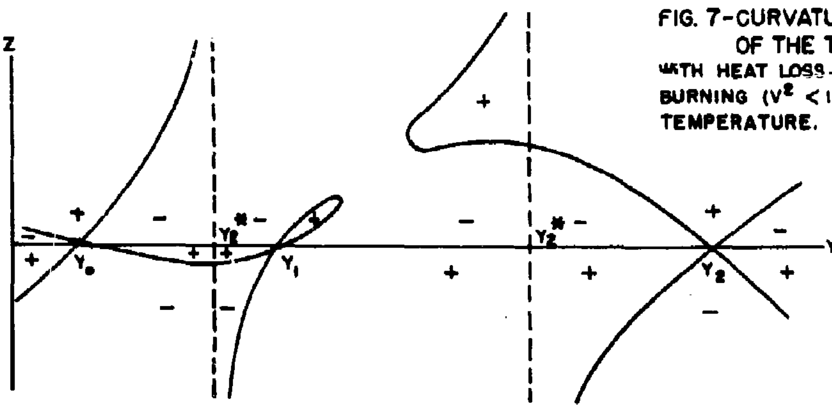


FIG. 7-CURVATURE OF INTEGRAL CURVES OF THE TRANSFORMED EQUATION. WITH HEAT LOSS- SMALL VELOCITY OF BURNING ($V^2 < 16e^{-2} - 4c$) LOW IGNITION TEMPERATURE.

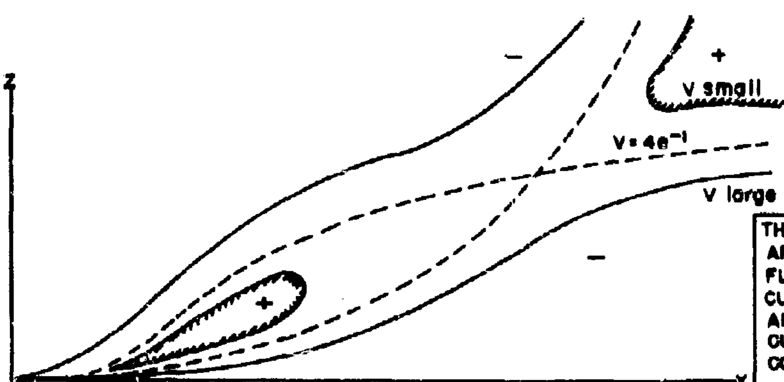


FIG. 8-CURVATURE OF INTEGRAL CURVES OF THE TRANSFORMED EQUATION. NO HEAT LOSS-VARIOUS VELOCITIES OF BURNING.

THE CURVES SHOWN IN FIG'S 6 - 8 ARE THE LOCI OF POINTS OF INFLECTION OF THE INTEGRAL CURVES. THE SIGNS (+ OR-) DENOTE AREAS WHERE THE INTEGRAL CURVES ARE CONCAVE OR CONVEX RESPECTIVELY.

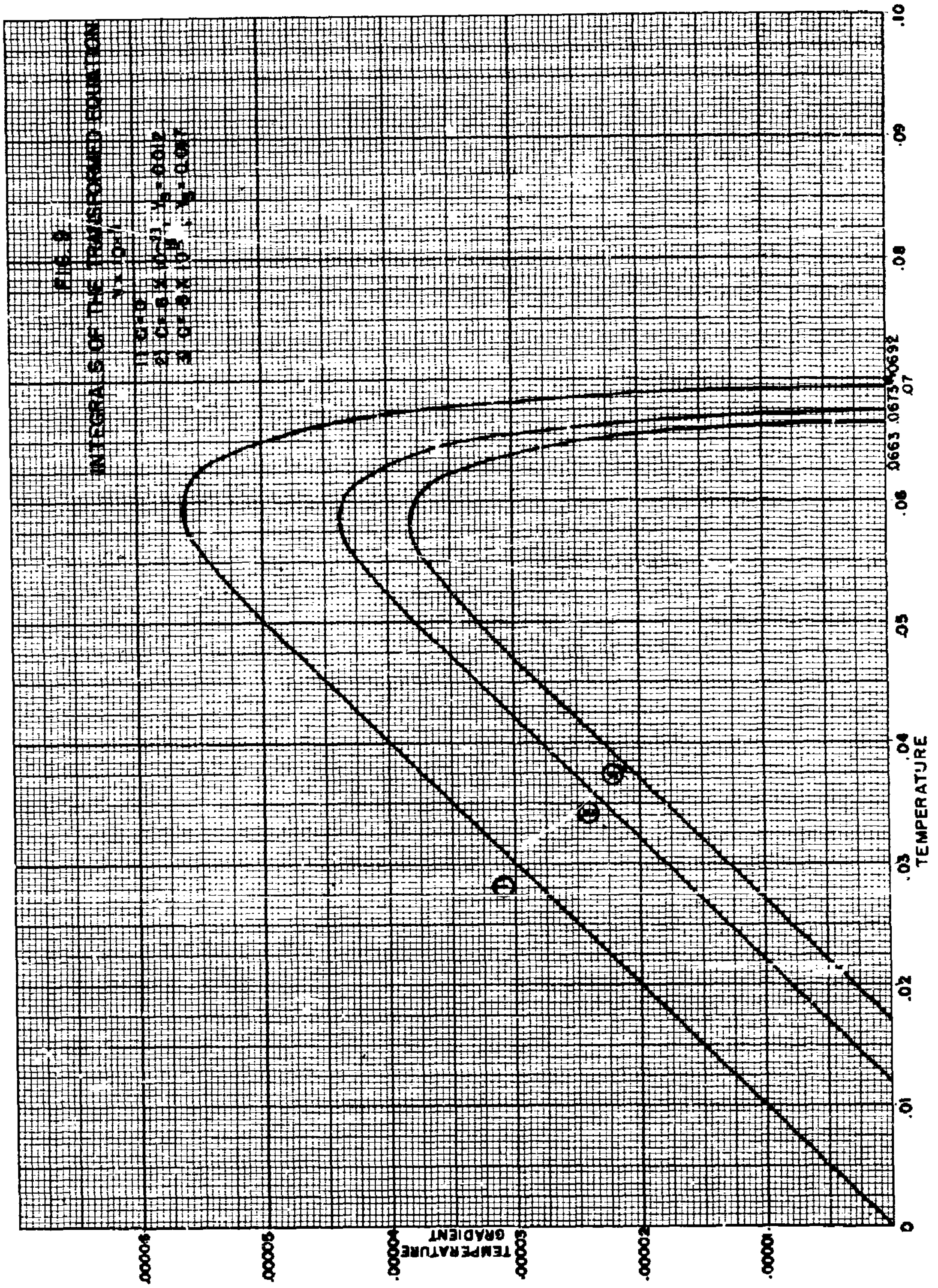


FIG. 9
INTEGRALS OF THE TRANSFORMED EQUATION
TEMPERATURE
TEMPERATURE GRADIENT
0
.01
.02
.03
.04
.05
.06
.0663
.0673
.0692
.07
.08
.09
.10
.00001
.00002
.00003
.00004
.00005
.00006

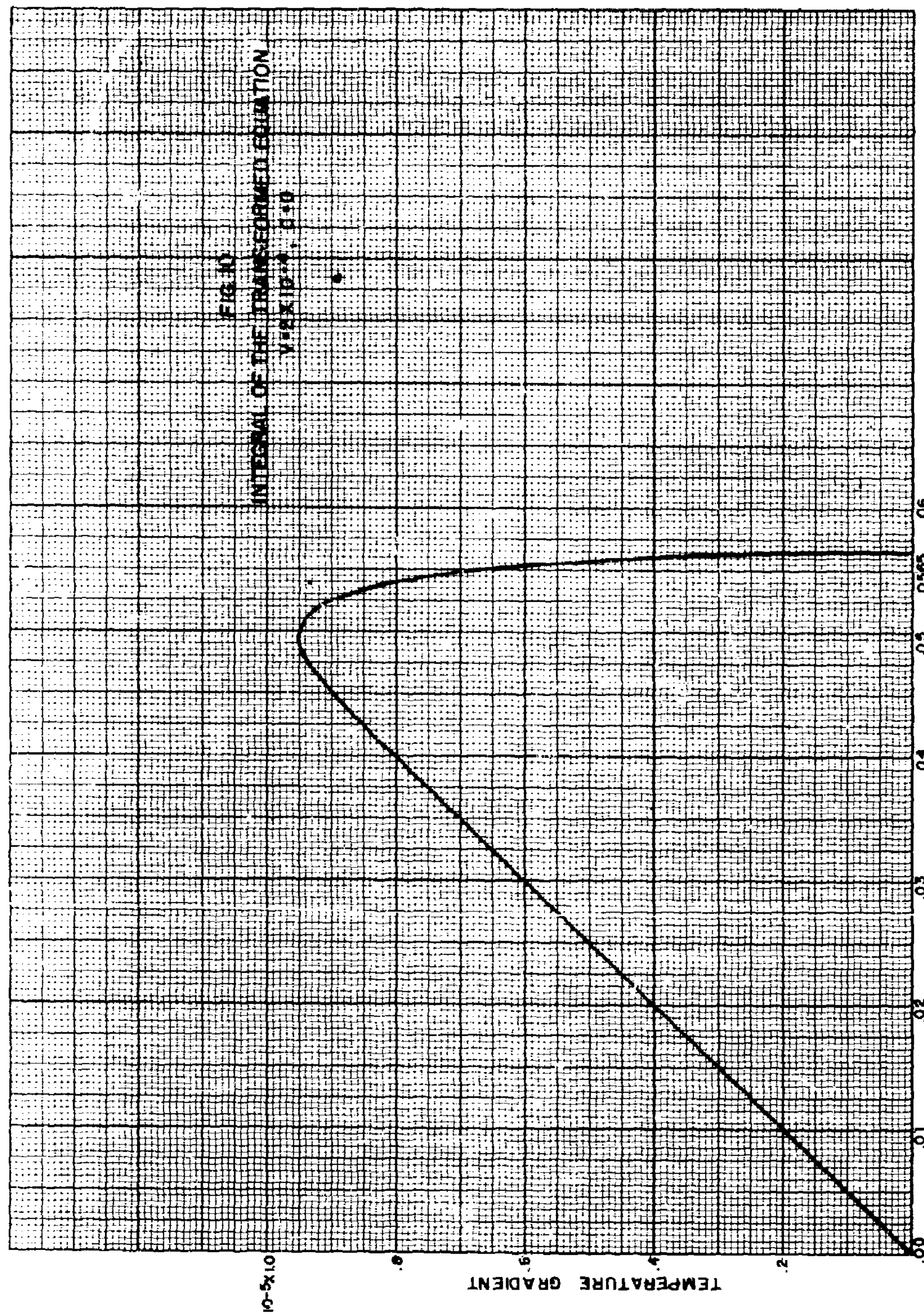


FIG. 10
INTEGRAL OF THE TRANSFORMED EQUATION
V=8X10^-4, D=0
TEMPERATURE
.00
.01
.02
.03
.04
.05
.0565
.06
TEMPERATURE GRADIENT
10^-5 X 1.0
.8
.6
.4
.2

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